Exam 2

Thu, Oct. 30

8:00 – 9:15

Arrive at least 5 min early!

Stepan Center

Format:

11 multiple choice questions

(Only 10 (best) counts)

3 free response questions

Practice exams on Canvas

The exam will cover:

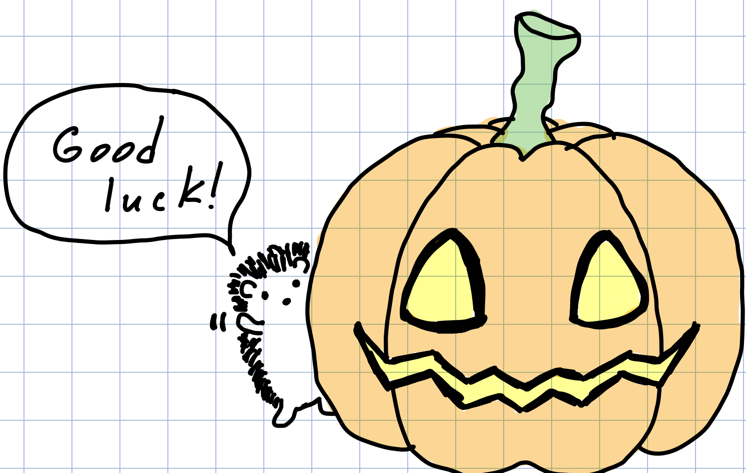
Chapter 14

and Sections 15.1, 15.2

Lectures 11 – 22

See also Review 4 & 5

Calculators: NOT allowed



Notes:

- Limits and Continuity
- Chain Rule (and Implicit differentiation)
- Review 4
- Review 5
- Lectures 11-22

(Lecture 11)

Functions of Several Vareadles and level curves
Partial derivatives (Lectures 12, 14)
Directional derivatives and gradient (Lect. 15,16)

Review 4

Local Extrema (Lect. 17)
Max/min on bounded regions (Lect. 18)
Lagrange multipliers (Lect. 19, 20)
Double integrals (Lect. 21, 22)

Review 5

1) What is the geometrical meaning of

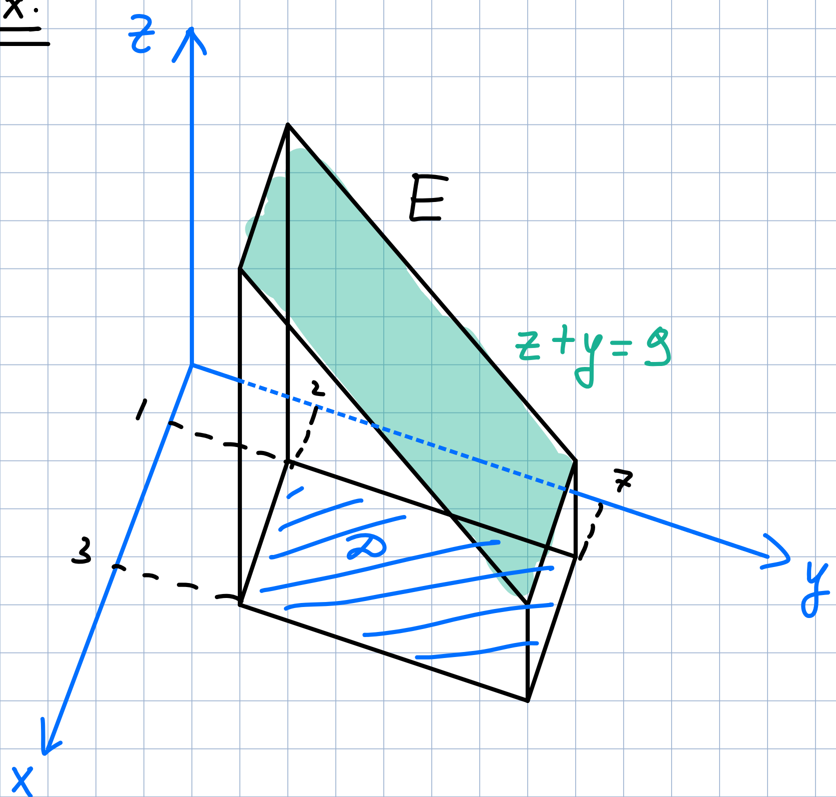
$$\iint_{D} 1\, dA \;?$$

A: $\iint_{D} 1\, dA = \text{Area}(D)$

2) If $f \geq 0$ what is the geometrical meaning of

$$\iint_{D} f(x,y)\, dA \;?$$

Ex:



Find the volume of E.

$$\iint_{D} f(x,y)\, dA = \int_1^3 \int_2^7 (9-y)\, dy\, dx$$

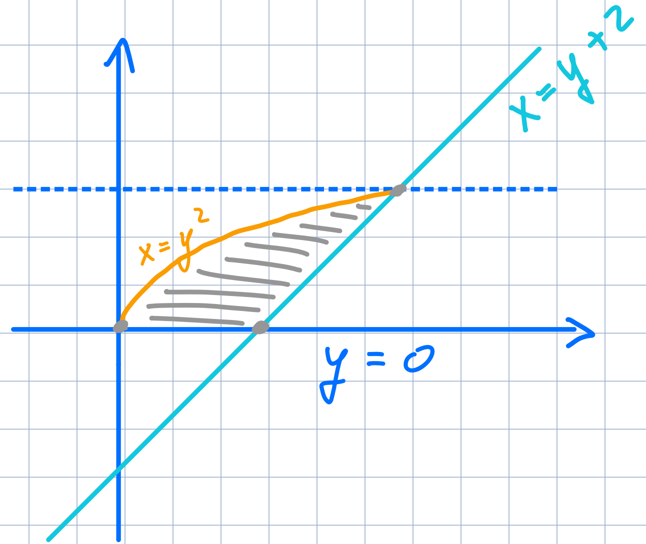
// $z+y=9 \iff z=9-y \qquad f(x,y)=9-y$ //

$$\text{Volume}(E) = \int_1^3 \int_2^7 (9-y)\, dy\, dx$$

Ex: Let $\mathcal{D}$ be a region bounded by $y=0$ on the bottom, $x=y^2$ on top and $x=y+2$ on the right. Rewrite $\iint_{\mathcal{D}} f(x,y)\,dA$ as an iterated integral.

Sol:

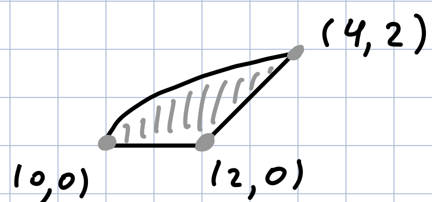
1) Sketch $\mathcal{D}$

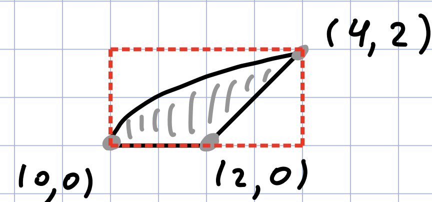


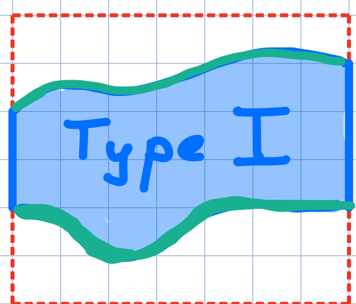
$\begin{cases} y=0 \\ x=y+2 \end{cases}$ $\quad\begin{cases} x=y^2 \\ y=0 \end{cases}$ $\quad\begin{cases} x=y+2 \\ x=y^2 \end{cases}$ & $y\geq 0$

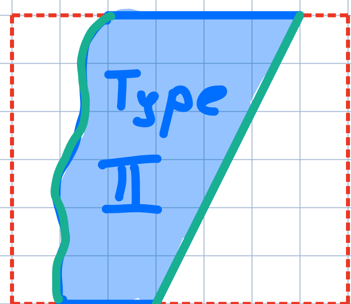
$(2,0)$ $\quad(0,0)$ $\quad(4,2)$

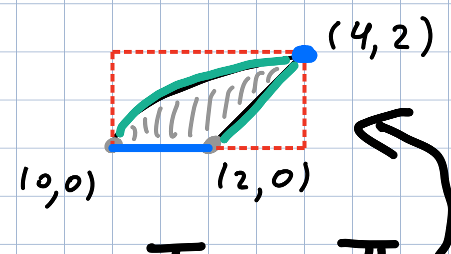
2) Type I VS Type II





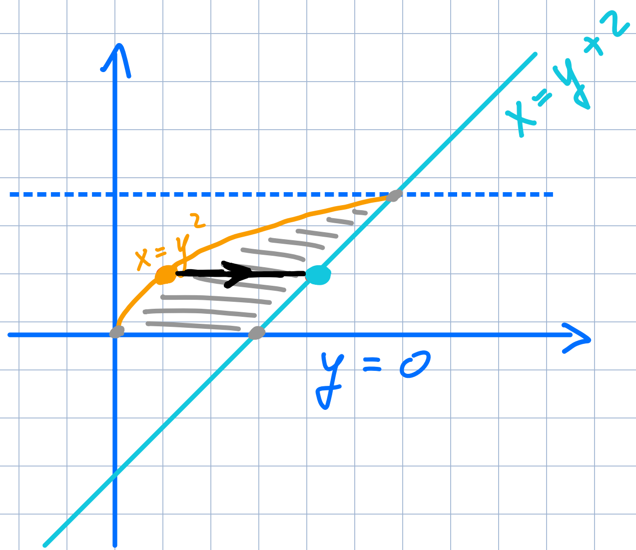


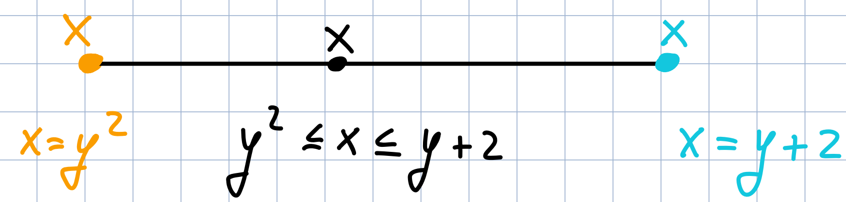




Type II

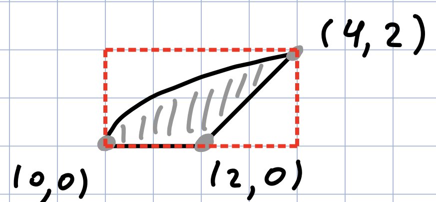
3) $\mathcal{D} = \{ (x,y) \mid 0 \le y \le 2,\ y^2 \le x \le y+2 \}$

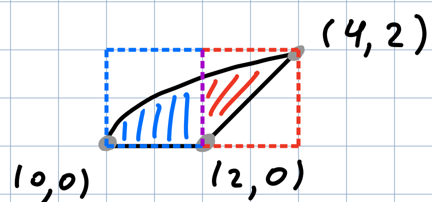




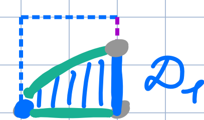
4) $\displaystyle\iint_{\mathcal{D}} f(x,y)\,dA = \int_0^2 \int_{y^2}^{y+2} f(x,y)\,dx\,dy$

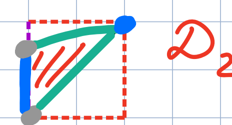
Q: Could we change the order of integration? Yes!





$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$





$\mathcal{D}_1: \quad 0 \le x \le 2, \quad 0 \le y \le \sqrt{x}$

$\mathcal{D}_2: \quad 2 \le x \le 4, \quad x-2 \le y \le \sqrt{x}$

$$\int_0^2 \int_{y^2}^{y+2} f(x,y)\,dx\,dy = \overbrace{\int_0^2 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx}^{\mathcal{D}_1} + \overbrace{\int_2^4 \int_{x-2}^{\sqrt{x}} f(x,y)\,dy\,dx}^{\mathcal{D}_2}$$

For more examples see Review 5

True-False

1) $f(x,y,z)$ $\quad \nabla f = \langle f_x, f_y \rangle$ F

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$\nabla h(x,y,z) = \langle h_x(x,y,z), h_y(x,y,z), h_z(x,y,z) \rangle$

2) $f(x,y) = e^x + xy$, $\vec{v} = \langle 3, 4 \rangle$ then the directional derivative of f in the direction of $\vec{v}$ is $3(e^x + y) + 4x$ F

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ $\quad \nabla f = \langle f_x, f_y \rangle = \langle e^x + y, x \rangle$ $\quad D_{\vec{u}} f = \vec{u} \cdot \nabla f = \frac{3}{5}(e^x + y) + \frac{4}{5}x$

3) $\nabla f(x,y)$ is perpendicular to the surface $z = f(x,y)$ F

$F(x,y,z) = f(x,y) - z$ $\quad F(x,y,z) = 0 \iff z = f(x,y)$

$\nabla F = \langle f_x, f_y, -1 \rangle$ perp. to level surf.

4) What is the direction of max decrease of $f(x,y,z)$?

$-\nabla f = \langle -f_x, -f_y, -f_z \rangle$ with the rate $|\nabla f|$.

5) How to find tangent plane/normal line to $z = f(x,y)$?

$F(x,y,z) = f(x,y) - z$, consider level surface $F(x,y,z) = 0$

∇F is a direction vector for the line and normal vector for the plane.

$f(x,y)$ - continuous function.

1) What can we say about $f(x_0,y_0)$ if $\nabla f(x_0,y_0)=0$?

(x_0,y_0) is a critical point,
f has a local max/min or a saddle point at (x_0,y_0)

2) What is the difference between

(Find max/min values of $f(x,y)$ subject to constraint $g(x,y)=k$) AND (Find max/min of $f(x,y)$ on bounded regions)?

Ex:

[Find extreme values of $f(x,y)=x^2+2y^2$ on the circle $x^2+y^2=1$.] ← subject to constraint

max/min on bounded region → [Find extreme values of $f(x,y)=x^2+2y^2$ on the disk $x^2+y^2\le 1$]

See Lecture 19 for solutions of both.

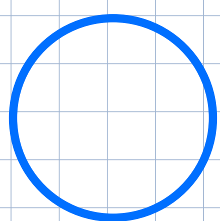
On the plane

- $\underbrace{g(x,y)=k}$ is a level curve of the function g

  describes a **CURVE**

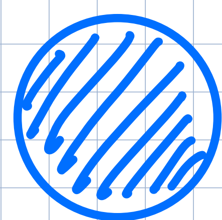
- the boundary of a region is a curve

Ex: $g(x,y) = x^2 + y^2$



circle

$g(x,y) = 1$



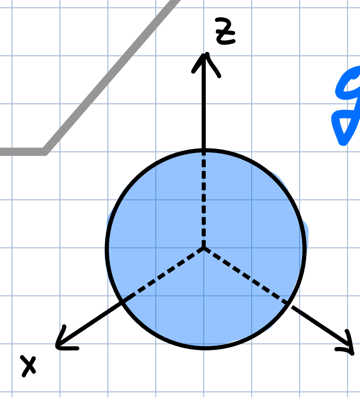
disk

$g(x,y) \leq 1$

In the space

- $g(x,y,z) = k$ is a level surface
- the boundary of a solid is a surface

Ex: $g(x,y,z) = x^2 + y^2 + z^2$



$g = 1$ sphere

$g \leq 1$ ball

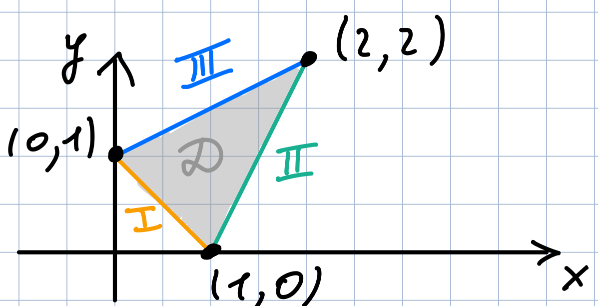
1) Find the points on the sphere $x^2+y^2+z^2=4$ closest to / farthest from the point $P(3,1,-1)$

2) The room has triangular shape with vertices at $(0,1)$, $(1,0)$ and $(2,2)$. The temperature is given by $t(x,y) = xy - 2x^2$. Where is the coolest place in the room?

(1) - subject to constraint [See Lecture 19]

(2) - min in the region

(2)



Step 1: Find critical point inside $\mathcal{D}$

$\nabla t = \langle y - 4x,\ x \rangle$

$$\begin{cases} y - 4x = 0 \\ x = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$(0,0)$ is NOT inside $\mathcal{D}$

Step 2: Find min values on the boundary

In order to do step 2 divide the boundary into 3 pieces and find min value at each piece of the boundary:

$x + y = 1$

$2x - y = 2$

$2y - x = 2$

I) $y = 1 - x$, $0 \le x \le 1$

$$\begin{cases} f(x,y) = f(x, 1-x) = x(1-x) - 2x^2 \\ 0 \le x \le 1 \end{cases}$$

$$\begin{cases} f(x, 1-x) = x - 3x^2 \\ 0 \le x \le 1 \end{cases}$$

WANTED: min of $h(x) = x - 3x^2$ on $[0, 1]$.

$h'(x) = 1 - 6x$ $\quad h'(\frac{1}{6}) = 0$

$h(0) = 0$

$h(1) = -2$ $\leftarrow$ min

$h(\frac{1}{6}) = 1/12$

$f(1, 1-1) = f(1, 0) = -2$

II) $y = 2x - 2$, $1 \le x \le 2$

$$\begin{cases} f(x,y) = f(x, 2x-2) = x(2x-2) - 2x^2 \\ 1 \le x \le 2 \end{cases}$$

$$\begin{cases} f(x, 2x-2) = -2x \\ 1 \le x \le 2 \end{cases}$$

WANTED: min of $h(x) = -2x$ on $[1, 2]$

$h(1) = -2$ $\quad h(2) = -4$ $\leftarrow$ min

$f(2, 2 \cdot 2 - 2) = f(2, 2) = -4$

III) $y = \frac{1}{2}(2+x)$, $0 \le x \le 2$

$$\begin{cases} f(x,y) = f\left(x, \frac{1}{2}(2+x)\right) = x\,\frac{2+x}{2} - 2x^2 \\ 0 \le x \le 2 \end{cases}$$

$$\begin{cases} f\left(x, \frac{2+x}{2}\right) = x - \frac{3}{2}x^2 \\ 0 \le x \le 2 \end{cases}$$

WANTED: min of $h(x) = x - \frac{3}{2}x^2$ on $[0,2]$

$h'(x) = 1 - 3x \qquad h'\left(\frac{1}{3}\right) = 0$

$h(0) = 0$

$h(2) = -4 \quad \leftarrow$ min

$h\left(\frac{1}{3}\right) = \frac{1}{6}$

$$f\left(2, \frac{2+2}{2}\right) = f(2,2) = -4$$

The smallest temperature is at the point $(2,2)$.

Ex: Find min of $f(x,y) = x$ subject to constraint $y^2+4x^2=1$.

Sol: $g(x,y) = y^2+4x^2$

$\nabla f = \langle 1, 0 \rangle$

$\nabla g = \langle 8x, 2y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases}$$

$$\begin{cases} 1 = \lambda(8x) \\ 0 = \lambda(2y) \\ y^2+4x^2=1 \end{cases}$$

1) IF $\lambda \neq 0$

$$\begin{cases} x = \frac{1}{8\lambda} \\ y = 0 \\ y^2+4x^2=1 \end{cases} \iff \begin{cases} x = \frac{1}{8\lambda} \\ y = 0 \\ 4\left(\frac{1}{8\lambda}\right)^2 = 1 \end{cases}$$

// $\lambda = \pm\frac{1}{4}$ //

$\lambda = \frac{1}{4} \Rightarrow x = \frac{1}{2}, y = 0$ $\quad f\left(\frac{1}{2}, 0\right) = \frac{1}{2}$

$\lambda = -\frac{1}{4} \Rightarrow x = -\frac{1}{2}, y = 0$ $\quad f\left(-\frac{1}{2}, 0\right) = -\frac{1}{2}$

2) IF $\lambda = 0$

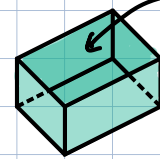
$$\begin{cases} 1 = 0(8x) \\ 0 = 0(2y) \\ y^2+4x^2=1 \end{cases}$$

$\leftarrow$ Never holds $\Rightarrow \lambda \neq 0$

The minimum is at the point $\left(-\frac{1}{2}, 0\right)$ and equal to $-\frac{1}{2}$.

Ex: Rectangular box without a lid is made of $12m^2$ of cardboard. What is the max possible volume?

Sol: $x, y, z > 0$ $\qquad f(x,y,z) = xyz$ – volume



$$g(x,y,z) = \underbrace{xy + 2xz + 2yz}_{\text{surface area}} = 12$$

Single constraint!

$$\begin{cases} yz = \lambda(y + 2z) \xrightarrow{\cdot \frac{1}{yz}} 1 = \lambda\left(\frac{1}{z} + \frac{2}{y}\right) & (1) \\ xz = \lambda(x + 2z) \xrightarrow{\cdot \frac{1}{xz}} 1 = \lambda\left(\frac{1}{z} + \frac{2}{x}\right) & (2) \\ xy = \lambda(2x + 2y) \xrightarrow{\cdot \frac{1}{xy}} 1 = \lambda\left(\frac{2}{y} + \frac{2}{x}\right) & (3) \\ xy + 2xz + 2yz = 12 \end{cases}$$

Note that $\lambda \neq 0$

(1) – (2): $2\lambda\left(\frac{1}{y} - \frac{1}{x}\right) = 0 \Rightarrow y = x$ $\qquad 1 = \lambda\left(\frac{1}{z} + \frac{2}{y}\right) \Rightarrow 1 = \lambda \frac{4}{x} \Rightarrow \lambda = \frac{x}{4}$

(1) – (3): $\lambda\left(\frac{1}{z} - \frac{2}{x}\right) = 0 \Rightarrow z = \frac{x}{2}$ $\qquad xy + 2xz + 2yz = 12 \Rightarrow 3x^2 = 12$

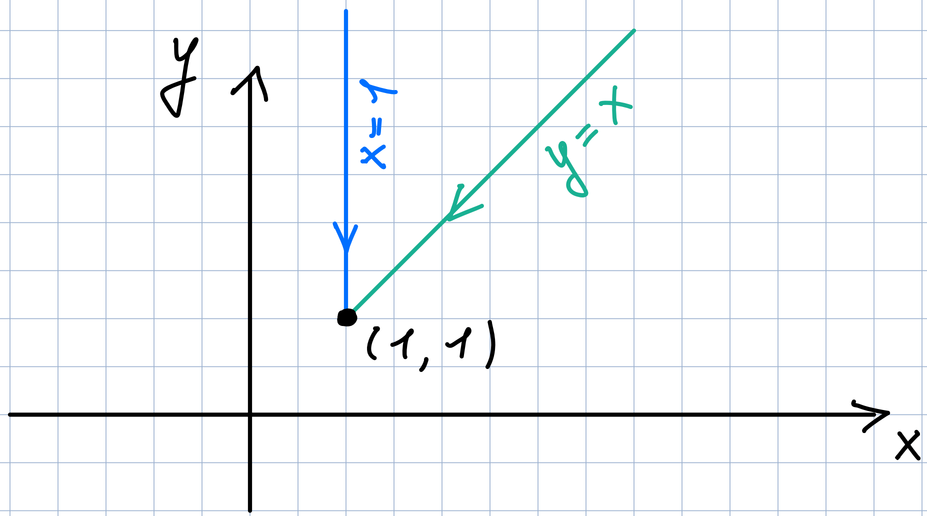
$\Rightarrow x = \pm 2$

should be $> 0 \Rightarrow 2$

$$\boxed{\begin{array}{l} x = 2, \quad y = 2, \quad z = 1 \\ \text{max volume: } \quad 2 \cdot 2 \cdot 1 = 4 \end{array}}$$

Ex: Show that $\lim_{(x,y)\to(1,1)} \frac{(x-1)^2}{(y-1)^2}$ does NOT exist.

Sol:



$$F(x,y) = \frac{(x-1)^2}{(y-1)^2}$$

Approaching along $y = x$:

$$F(x,x) = \frac{(x-1)^2}{(x-1)^2} = 1$$

Approaching along $x = 1$:

$$F(1,y) = \frac{(1-1)^2}{(y-1)^2} = 0$$

$0 \neq 1 \Rightarrow$ the limit does NOT exist.

Ex: Find a max/min value of $f(x,y,z) = x - z$ on the intersection of the cylinder $x^2+y^2=1$ and the plane $x+2z=0$.

Sol: 1) $x^2+y^2=1$ ← constraints → $g(x,y,z) = x^2+y^2$

$x+2z=0$ ← constraints → $h(x,y,z) = x+2z$

WANTED: Find max/min of $f(x,y,z)$ subject to two constr.

$$g(x,y,z) = 1$$
$$h(x,y,z) = 0$$

2) Apply Lagrange multipliers method:

We want to find all x, y, z, λ and μ such that

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = 1 \\ h(x,y,z) = 0 \end{cases}$$

3) Find the gradiants.

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 1, 0, -1 \rangle \qquad \nabla g = \langle 2x, 2y, 0 \rangle \qquad \nabla h = \langle 1, 0, 2 \rangle$

4)
$$\begin{cases} 1 = \lambda(2x) + \mu(1) \\ 0 = \lambda(2y) + \mu(0) \\ -1 = \lambda(0) + \mu(2) \\ x^2 + y^2 = 1 \\ x + 2z = 0 \end{cases}$$

$$\begin{cases} 1 = 2\lambda x + \mu \\ 0 = 2\lambda y \\ -1 = 2\mu \quad \Rightarrow \mu = -\frac{1}{2} \\ x^2 + y^2 = 1 \\ x + 2z = 0 \end{cases}$$

IF $\lambda \neq 0$:

$$\begin{cases} x = \frac{1-\mu}{2\lambda} \\ 0 = 2\lambda y \quad \Rightarrow y = 0 \\ \mu = -\frac{1}{2} \\ x^2 + y^2 = 1 \\ x + 2z = 0 \end{cases}$$

$$\begin{cases} \mu = -\frac{1}{2} \\ y = 0 \\ x = \frac{1-(-\frac{1}{2})}{2\lambda} = \frac{3}{4\lambda} \\ \left(\frac{3}{4\lambda}\right)^2 + 0^2 = 1 \\ \frac{3}{4\lambda} + 2z = 0 \end{cases}$$

$$\left(\frac{3}{4\lambda}\right)^2 = 1 \iff \lambda = \pm \frac{4}{3}$$

IF $\lambda \neq 0$

$\mu = -\frac{1}{2}$ $\quad y = 0$ $\quad \lambda = \frac{4}{3}$ $\quad x = 1$ $\quad z = -\frac{1}{2}$ $\quad f(1, 0, -\frac{1}{2}) = \frac{3}{2}$

$\mu = -\frac{1}{2}$ $\quad y = 0$ $\quad \lambda = -\frac{4}{3}$ $\quad x = -1$ $\quad z = \frac{1}{2}$ $\quad f(-1, 0, \frac{1}{2}) = -\frac{3}{2}$

If $\lambda = 0$:

$$\begin{cases} 1 = 2(0)x + \mu \\ 0 = 2(0)y \\ \mu = -\frac{1}{2} \\ x^2+y^2 = 1 \\ x + 2z = 0 \end{cases}$$

$\Rightarrow \mu = 1$

$\Rightarrow \mu = -\frac{1}{2}$

$\Rightarrow \lambda$ can NOT be zero.

So the max value is $\frac{3}{2}$ attained at $(1, 0, -\frac{1}{2})$

the min value is $-\frac{3}{2}$ attained at $(-1, 0, \frac{1}{2})$